

Dilaton spacetimes with a Liouville potential

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Abstract

We find and study solutions to the Einstein equations in D dimensions coupled to a scalar field source with a Liouville potential under the assumption of $D - 2$ planar symmetry. The general static or time-dependent solutions are found yielding three classes of $SO(D - 2)$ symmetric spacetimes. In $D = 4$ homogeneous and isotropic subsets of these solutions yield planar scalar field cosmologies. In $D = 5$ they represent the general static or time-dependent backgrounds for a dilatonic wall-type brane Universe of planar cosmological symmetry. Here we apply these solutions as $SO(8)$ symmetric backgrounds to non-supersymmetric 10 dimensional string theories, the open $USp(32)$ type I string and the heterotic string $SO(16) \times SO(16)$. We obtain the general $SO(9)$ solutions as a particular case. All static solutions are found to be singular with the singularity sometimes hidden by a horizon. The solutions are not asymptotically flat or of constant curvature. The singular behavior is no longer true once we permit space and time dependence of the spacetime metric much like thick domain wall or global vortex spacetimes. We analyze the general time and space dependent solutions giving implicitly a class of time and space dependent solutions and describe the breakdown of an extension to Birkhoff's theorem in the presence of scalar matter. We argue that the solutions described constitute the general solution to the field configuration under $D - 2$ planar symmetry.

I. INTRODUCTION

The dimensional nature of the manifold we live in has been an intriguing question for mathematicians and physicists alike in the past century. Minkowski made the first crucial step [1] in this direction introducing time as part of a 4 dimensional space-time manifold. Of course time, at every day low velocities, is not perceived as a coordinate, however, at accelerator velocities close to the speed of light, time and length can vary, the prefactor c^2 acting

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as a dimensional ‘warp-like’ factor in Einstein’s theory of special relativity. Furthermore, with the introduction of timelike coordinates in general relativity, dynamics are encoded in spacetime geometry. This geometrical idea, embodied in Mach’s principle, was taken further by Kaluza and Klein who obtained electromagnetism as part of the geometry of a *vacuum* 5-dimensional spacetime rather than a source term of the 4 dimensional Einstein equations (see for example [2] for a general discussion on Kaluza-Klein gravity).

In recent years string theory has been the main advocate of higher dimensional theories but also independently in the early 80’s brane Universe models were introduced [3], and also simple models where the 5th dimension was *dynamically* compactified by a time-like contraction of the 5th dimension in a 5 dimensional Kasner vacuum solution [4]. The subject was revived with Hörawa-Witten [5] cosmology, related works [6], [7], [8] and in the last two years with the brane-Universe models proposed in [9], [10].

It is reasonable to argue from the point of view of Unified theories that any additional dimensions of spacetime must in some way be connected with string theory where gravity is unified with the fundamental elementary interactions. One would like to ultimately bridge our knowledge of standard cosmology and the Standard Model to the theoretical realm of string theory. However on going to higher dimensions one does not want to violate rather precise experimental data of standard model physics and furthermore lose the unique characteristics of a 4 dimensional spacetime. Renormalisability of gauge interaction and non-trivial gravity, with the graviton acquiring two polarization degrees of freedom, agree only for a 4 dimensional spacetime (for a discussion see [11]). Indeed the former constraint is often partially embodied in the assumption that the Standard Model fields do not *see* the extra dimensions being strictly confined on a 4 dimensional ‘braneworld’. Of course any fields weakly interacting with the Standard Model have no reason to obey such a restriction. Gravity or closed strings in particular see the extra dimensions by definition. Fermionic matter such as sterile neutrinos can also propagate in the extra dimensions and models [12] have been introduced to explain experimental data of neutrino mass oscillations (see [13] for a recent review on neutrino physics). Note however that a unique sterile neutrino is less favored by the recent experimental data [14]. Furthermore cold dark matter particles such as axions or WIMPS, which in some cases can be modeled by a scalar field and a self-interaction potential, could also constitute such bulk matter. These rather general and speculative observations are dictated to us by TeV scale physics.

At the theoretical realm of beyond the standard model physics, bulk scalar matter originates naturally from string theory in the guise of the dilaton field. It is such scalar matter, as a source to the Einstein equations that we shall be considering here. In supersymmetric string theories the dilaton potential is ‘protected’ by supersymmetry. Hence a dilaton field as matter source corresponds to the rather unphysical case of a stiff perfect fluid, where pressure is equal to energy density and the velocity of sound is the velocity of light. The breaking of spacetime supersymmetry in 10 dimensions however, results in the appearance of a dilaton tadpole, which boils down quite generically to a Liouville type potential in the field content of the classical low energy theory. Furthermore as a result spacetime does not admit solutions of maximal symmetry in particular a Minkowski, de-Sitter or AdS background. Solutions of lesser, $SO(9)$ symmetry, have to be found [15] and rather interestingly spacetime can have a maximally symmetric 9-dimensional behavior, the 10th dimension being compact. Furthermore from a cosmology induced perspective the energy-momentum

tensor in the presence of the potential can be sometimes treated as a perfect fluid source with energy-density ρ and pressure P given by,

$$\rho = -\frac{1}{4}\partial_\mu\phi\partial_\nu\phi g^{\mu\nu} + V(\phi) \quad P = -\frac{1}{4}\partial_\mu\phi\partial_\nu\phi g^{\mu\nu} - V(\phi)$$

Of course no equation of state is specified for $V \neq 0$.

In this paper we shall find solutions to the Einstein equations coupled with a scalar field and a self-interaction Liouville potential, namely we shall consider a classical action of the form,

$$S_E = \frac{1}{2k^2} \int d^D x \sqrt{-G} [R - \frac{1}{2}(\partial\Phi)^2 - 2\alpha e^{\gamma\Phi}], \quad (1)$$

We shall carry out our analysis in an arbitrary number of dimensions D . We shall consider spacetimes admitting $D - 2$ planar spacelike surfaces the exact cosmological setting for a planar dilatonic domain wall brane Universe in $D = 5$ first discussed in [7] (see also [16], [17], [18]). Here we shall be applying our $SO(8)$ solutions in $D = 10$ tachyon free, 10 dimensional non-supersymmetric string theories. The possibility of having anomaly free non supersymmetric string theories was noted in [19]. Tachyon free non-supersymmetric models were constructed in [20] and more recently in [21] and [22] (for a recent discussion on anomaly related issues see [23]). Here we shall apply our solutions to the open type I string with gauge group $USp(32)$ [22] and the $SO(16) \times SO(16)$ closed heterotic string [20] with cosmological constant. The dilaton tadpole is portrayed by the Liouville potential in (1) with yields $\gamma = 3/2$ for the open string and with $\gamma = 5/2$ for the heterotic string to leading order in the string coupling expansion.

On considering solutions with $SO(9)$ symmetry it is found that the presence of the dilaton induces naked singularities of spacetime where the low energy classical theory breaks down [15]. One can question the persistence of this singularity if the symmetry of spacetime is relaxed to 8-dimensional Poincaré symmetry. Indeed topological defect solutions such as domain walls [24], [25] are not singular (away from the distributional source of course) once spacetime is allowed to be time dependent. In complete analogy planar thick domain walls where matter is described by a real scalar field in a typical double well potential are everywhere smooth and non-static (see [26] and references within). Global vortices [27] are in analogy non-singular once we allow the spacetime metric to be time dependent. We will see a similar property arising here (see also in this context the recent work of Lidsey [28]).

Also the question of asymptotic flatness is an important issue. As we noted no maximally symmetric solutions exist for these models. However one would expect that the dilaton would roll down the potential well assuming its vacuum value for ϕ going to minus infinity. It is precisely the fact that the dilaton acquires its minimum value at a non-finite value which yields all solutions not asymptotically flat or even of constant curvature. This is a characteristic of exponential potentials investigated in [29] for the case of spherically symmetric black holes (for a more general discussion on properties of massive dilatonic black holes see [30]).

Furthermore in a more mathematical frame or mind another question arises : Does a generalization of Birkhoff's theorem hold in the presence of a scalar field source with a Liouville potential? Hence does spacetime admit an extra timelike or spacelike Killing vector

reducing all possible solutions to being locally static *or* time-dependent? We will see here that this is not the case where a simple concrete example was provided recently in [18] (see also [17]). On analyzing the two dimensional solutions we will see exactly how Birkhoff's theorem breaks down in the case of a scalar field.

In what follows we shall find the general static or time dependent solutions i.e. solutions where our fields depend on a spacelike or timelike independent variable uniquely. We call these solutions *one dimensional* solutions. We shall also give implicitly some *two-dimensional* solutions analyzing a simple example of these [18] and analyze in detail the field equations in the presence or not of a Liouville potential.

Three classes of one dimensional solutions emerge, distinguished by the discriminant of a second degree polynomial $f(p)$ and depending essentially on two integration numerical constants c and d . The constant c will take three values $\pm 1, 0$ and will characterize the topology of the solutions. The constant d will be associated to the Weyl curvature of spacetime and in the presence of a black hole horizon will represent the quasilocal mass (see for example [31] and references within). In turn vanishing d and hence Weyl tensor, will yield the general $SO(9)$ symmetric solutions which will be the maximal symmetry solutions for our field set-up.

The roots of the polynomial $f(p)$ will always yield coordinate or naked singularities of spacetime. Class I solutions for example, the two distinct root case, will admit timelike and spacelike solutions, as for example for $d = 0$ the general $SO(9)$ solutions for the heterotic string: we will find that there exist three spacelike solutions including the static [15] solution and one timelike solution, the $D = 10$ dimensional version of scalar field cosmology solutions in $D = 4$ [32] recently revived in the context of quintessence (see for example [33]). For Class II exactly half of the solutions will be compact in the p direction and regular for finite p . The other half will be singular at the origin and non-compact. Class III solutions will be always compact. In particular we will find that the Type I open string holds a particular singular position and will have to be treated separately yielding completely different results.

A general characteristic of all static solutions is that they are of finite proper distance i.e. of $M^9 \times S^1/Z_2$ topology if and only if the curvature tensor is singular at the endpoints of the interval.

We start in the next section by setting up the field equations we shall study. We then in section III extend the method of [34] and discuss black hole type solutions. In section IV and V we give the general static or time dependent solutions for an $SO(D - 2)$ symmetric spacetime. We analyze in detail the full equations in section VI, giving implicit 2-dimensional solutions and also give a sketch of the general two dimensional solution in the absence of the potential ($\alpha = 0$). We summarize our results and conclude in section VII.

II. GENERAL SET-UP

In this section we shall give the formal setup of the field equations for an arbitrary number of spacetime dimensions D . Consider the classical theory described by the following effective action (written in the Einstein frame),

$$S_E = \frac{1}{2k^2} \int d^D x \sqrt{-g} [R - \frac{1}{2}(\partial\Phi)^2 - 2\alpha e^{\gamma\Phi}], \quad (2)$$

where α and γ are positive constants of our theory. For string theory in $D = 10$ they are related to the string tension and the leading coefficient in the string coupling expansion respectively. For example for the $Usp(32)$ Type I string $\alpha = 64k^2T_9$ where T_9 is the positive tension of the space filling \bar{D}_9 brane and $\gamma = 3/2$ is derived by the disc and projective plane amplitudes of the open string theory¹.

Let us consider a space-time metric admitting a $(D - 2)$ -dimensional planar spacelike surface. A general metric admitting this symmetry can be written [35],

$$ds^2 = e^{2\nu} B^{-\frac{D-3}{D-2}} (-dt^2 + dz^2) + B^{\frac{2}{D-2}} dx_8^2 \quad (3)$$

where ν and B are functions of a timelike coordinate t and spacelike coordinate z . Note that metric (3) is the typical bulk setup for brane Universe cosmology [34]. Our source is a scalar field $\phi = \phi(t, z)$ with a Liouville self-interaction potential given by the matter Lagrangian and energy-momentum tensor read off from (2)

$$\mathcal{L}_{\mathcal{M}} = -\frac{1}{2} \partial_\lambda \phi \partial^\lambda \phi - 2\alpha e^{\gamma\phi} \quad (4a)$$

$$T_{ab} = \frac{1}{2} \partial_a \phi \partial_b \phi + \frac{1}{2} g_{ab} \mathcal{L}_{\mathcal{M}} \quad (4b)$$

We shall seek solutions to the coupled Einstein and scalar field equations for metric (3) and dilaton field matter (4) in D dimensions,

$$R_a^b = T_a^b - \frac{1}{D-2} \delta_a^b T \quad (5)$$

$$\square \phi - 2\alpha \gamma e^{\gamma\phi} = 0. \quad (6)$$

After some reshuffling the field equations take the form,

$$B_{tt} - B_{zz} = 2\alpha B^{\frac{1}{D-2}} e^{2\nu+\gamma\phi} \quad (7a)$$

$$\nu_{tt} - \nu_{zz} = \frac{\alpha}{D-2} B^{-\frac{D-3}{D-2}} e^{2\nu+\gamma\phi} + \frac{1}{4} (\phi_z^2 - \phi_t^2) \quad (7b)$$

$$\phi_{tt} - \phi_{zz} = -2\alpha \gamma B^{-\frac{D-3}{D-2}} e^{\gamma\phi+2\nu} + \phi_z \frac{B_z}{B} - \phi_t \frac{B_t}{B} \quad (7c)$$

$$2\nu_z B_t + 2\nu_t B_z - B_{tz} = B \phi_t \phi_z \quad (7d)$$

$$2\nu_z B_z + 2\nu_t B_t - B_{tt} - B_{zz} = \frac{B}{2} (\phi_t^2 + \phi_z^2) \quad (7e)$$

The metric components (3) are expressed in such a way so that equations (7a), (7b) and (7c) are non-homogeneous wave equations with respect to B , ν and ϕ for all D whereas (7d) and (7e) are interpreted as integrability equations for these. If $\gamma = 0$ and the scalar field is constant throughout spacetime, then the field equations reduce to the Einstein equations in the presence of a cosmological constant α . The general solution for this case has been found

¹Note that in principle we could be including the next order contribution ($\gamma = 5/2$) originating from the torus in the string coupling constant expansion. We shall comment on such a possibility in our conclusions

and treated in detail in [34]. In the same frame of mind it is useful to transform to light cone coordinates,

$$u = \frac{t - z}{2}, \quad v = \frac{t + z}{2} \quad (8)$$

upon which the field equations reduce to,

$$B_{uv} = 2\alpha B^{\frac{1}{D-2}} e^{2\nu + \gamma\phi} \quad (9a)$$

$$\nu_{uv} = \frac{\alpha}{D-2} B^{-\frac{(D-3)}{(D-2)}} e^{2\nu + \gamma\phi} - \frac{1}{4} \phi_u \phi_v \quad (9b)$$

$$\phi_{uv} = -2\alpha\gamma B^{-\frac{D-3}{D-2}} e^{\gamma\phi + 2\nu} - \frac{1}{2B} (\phi_u B_v + \phi_v B_u) \quad (9c)$$

$$2\nu_u - [ln(B_u)]_u = \frac{B}{2B_u} \phi_u^2 \quad (9d)$$

$$2\nu_v - [ln(B_v)]_v = \frac{B}{2B_v} \phi_v^2 \quad (9e)$$

a system of 3 non-homogeneous wave equations for B , ν and ϕ constrained by the *ordinary* differential equations (9d), (9e). It can be shown that (9b) is redundant resulting from the remaining equations, however, its form can be instructive and we keep it.

Before proceeding into the search for solutions to (9), let us note the two dimensional (in the $t - z$ plane) conformal symmetries,

$$u \rightarrow f(u) \quad w \rightarrow g(v) \quad (10)$$

where f and g are arbitrary functions which leave (3) invariant. This is an essential symmetry of the problem in the metric Ansatz we have chosen, which will be seen to reduce seemingly two-dimensional solutions, to solutions which are in fact one-dimensional. Also the form of the metric (3) dictates that by a Wick rotation a static one-dimensional solution will transform onto a time dependent one dimensional solution. We shall call one-dimensional solutions the ones that after a suitable coordinate transformation can be seen to depend (locally) on a timelike *or* a spacelike coordinate. Two dimensional solutions will be those depending on time *and* space such that there exists no coordinate transformation or equivalently no timelike or spacelike Killing vector reducing them to a one-dimensional solution.

III. DILATON BLACK HOLES

As a first approach let us construct solutions using the method of [34] which extend the topological black hole solutions [36]. The topological black hole solutions have been extensively studied in string theory (see [37] and references within) and also in brane Universe cosmology [38] the earliest application dating [39] in the context of the Randall-Sundrum [10] model. We will see in the next section how the dilatonic version of these solutions are part of a more general class of solutions.

The starting point are conditions (9d) and (9e) which are not integrable equations as they stand. This is the mathematical difficulty we will have to face throughout our analysis

and will lead to the breakdown of Birkhoff's unicity theorem. So choose a simple Ansatz that makes (9d) and (9e) integrable equations,

$$e^\phi = B^c e^{\phi_0} \quad (11)$$

with c and ϕ_0 real constants. On doing so, constraints (9d) and (9e) yield,

$$B = B(U(u) + V(v)), \quad e^{2\nu} = U'(u)V'(v)B'B^{c^2/2} \quad (12)$$

where U and V are arbitrary functions of a single variable and $'$ will always denote the derivative with respect to the unique argument of the function. On inputting (11), (12) in the wave equations of (9) we find that they are consistent for $\gamma \neq 0$ if and only if $c = -\gamma$. In order to simplify notation let us set

$$s = \frac{\gamma^2}{2} - \frac{D-1}{D-2}. \quad (13)$$

We will that the constant s plays a particularly important role, often determining the nature of the solutions. The case $c = \gamma = 0$ corresponds to the cosmological constant solution [36]. The wave equation (9a) for component B reduces to,

$$B' = -\frac{2\alpha}{s} B^{-s} e^{\gamma\phi_0} - d/2 \quad (14)$$

and in particular for $s = 0$

$$B' = 2\alpha e^{\gamma\phi_0} \ln B - d/2 \quad (15)$$

with d an arbitrary integration constant. Hence (3) admits the particular solution,

$$ds^2 = B'B^{\frac{\gamma^2}{2} - \frac{D-3}{D-2}} U'V'(-dt^2 + dz^2) + B^{\frac{2}{D-2}} dx_8^2 \quad (16)$$

for $s \neq 0$, with B' given by (14) and similarly for $s = 0$ with B' given by (15).

Now the important point is that this solution is a one dimensional solution. Indeed U and V reflect the conformal rescaling freedom (10) and are not physical degrees of freedom. To see this we fix (10) setting,

$$U = \frac{1}{2}(\bar{z} - \bar{t}), \quad V = \frac{1}{2}(\bar{z} + \bar{t}) \quad (17)$$

and hence $B = B(\bar{z})$ i.e. the solution is *static* for \bar{z} spacelike and vice-versa². Noting then that $d\bar{z} = \frac{dB}{B'}$ we get,

$$ds^2 = -B'B^{\frac{\gamma^2}{2} - \frac{D-3}{D-2}}(-d\bar{t}^2) + \frac{B^{\frac{\gamma^2}{2} - \frac{D-3}{D-2}}}{-B'} dB^2 + B^{\frac{2}{D-2}} dx_8^2 \quad (18)$$

²Alternatively take, $U = \frac{1}{2}(-\bar{z} + \bar{t})$, $V = \frac{1}{2}(\bar{z} + \bar{t})$ upon which $B = B(\bar{t})$.

for $s \neq 0$, with $-B'$ given by (14) as a function³ of B and accordingly for $s = 0$ using (15). The dilaton field is given by,

$$\phi = \phi_0 - \gamma \ln B$$

for all values of s . The form of these solutions, which are singular at $B = 0$, depends on the sign and zeros of B' . In fact as we can see from (14) and (15) depending on the sign of d and s , we will have black hole solutions or time dependent solutions with a cosmological horizon or again solutions with a naked singularity at $B = 0$. These solutions have been found and analyzed in a different coordinate system by Chan et al [31] for the spherical case and by Cai et al [40] for planar and hyperbolic geometry (see also recent work [41] for a general class of solutions). As backgrounds for the motion of dilatonic domain wall type Universes they were analyzed by Chamblin and Reall [7] and more recently in [18]. Since we are interested in the context of non-supersymmetric string theories we examine these solutions for $D = 10$ referring the reader to the above papers for further applications and properties of these solutions.

First of all we note that $\gamma = 3/2$ i.e. (13) $s = 0$, yields the critical value for the gravitational field (but not for the dilaton field), and corresponds to the Type I $USp(32)$ string. Hence whatever spacetime solutions we find for non-supersymmetric heterotic string theories, $\gamma = 5/2$, the Type I string solutions will be inherently different.

Let us start with the case of $\gamma = 5/2$ ($s = 2$), the non-supersymmetric heterotic string case. Then using (14), solutions (18) simplify to,

$$ds^2 = -\eta(-\alpha e^{5\phi_0/2} B^{-2} - d/2) B^{9/4} d\bar{t}^2 + \frac{B^{9/4} dB^2}{\eta(-\alpha e^{5\phi_0/2} B^{-2} - d/2)} + B^{1/4} dx_8^2 \quad (19)$$

where $\eta = \pm 1$, with

$$\phi = \phi_0 - \frac{5}{2} \ln B$$

Solutions (19) can be written in the string frame by the Weyl rescaling, $g_{\mu\nu}^{(S)} = e^{\phi/2} g_{\mu\nu}$. The $\eta = -1$ corresponds to (18) and the $\eta = 1$ to its counterpart. Hence for $\eta = 1$ if $d < 0$ then $-B'_H = \alpha e^{5\phi_0/2} B_H^{-2} + d/2 = 0$ corresponds to a cosmological horizon B_H and hence for the horizon exterior, B is a timelike coordinate with $B = 0$ a timelike singularity. Alternatively for $d > 0$, B is spacelike, and $B = 0$ is a naked timelike singularity. For the opposite sign if $d < 0$ then $B'_H = 0$ corresponds to a black hole horizon and hence for $B > B_H$, B is a spacelike coordinate and $B = 0$ is a spacelike singularity. The parameter d in this case is the quasilocal mass (see [31] and references within) of the black hole. For $d > 0$, B is timelike with $B = 0$ a naked spacelike singularity.

Let us now turn to the asymptotic behavior. Calculation of the Ricci scalar gives, $R = \frac{5}{8}(-9\alpha + 5 - d/2B^2)B^{-25/4}$ which ties in with the form of the dilaton potential. Note then that spacetime geometry coupled to matter is well behaved for large B , the Ricci scalar asymptoting zero. On the other hand note now the bizarre property of these solutions; their asymptotic behavior depends on the parameter d . Indeed for large B the solution asymptotes,

$$ds^2 = \eta dB^{9/4} d\bar{t}^2 + \frac{B^{9/4} 2dB^2}{-\eta d} + B^{1/4} dx_8^2$$

³For the timelike case simply replace $-B'$ by B' in (18)

and spacetime is not asymptotically flat although matter is vanishing at that region. Indeed as our coordinate B goes to infinity the dilaton field goes to minus infinity the dilaton potential acquiring thus its global minimum. Hence even though the scalar field rolls down the potential well spacetime is not trivial. The (partial) resolution of this discrepancy lies in the physical interpretation of d which is related to the Weyl tensor. Indeed setting $d = 0$ one can show that the Weyl tensor is identically zero. Hence we can deduce that although matter tends to the vacuum (and the Ricci tensor with it) the Weyl tensor i.e. pure curvature controls the large B region where spacetime exhibits purely gravitational tidal forces depending on the magnitude of d . So a far away observer for the dilatonic black hole is subject to gravitational tidal forces dependent on the quasilocal mass d , quite in contrast to the standard situation of a Schwarzschild black hole which is asymptotically flat regardless of its ADM mass.

Setting $d = 0$ we get solutions of maximal $SO(9)$ symmetry for the heterotic string, $\gamma = 5/2$ which were first discussed in [42]. The solution reads,

$$ds^2 = B^{1/4}[-\eta d\bar{t}^2 + dx_8^2 \frac{\eta B^4 dB^2}{\alpha e^{5\phi_0/2}}] \quad (20)$$

and exhibits a naked singularity at $B = 0$. Furthermore as we pointed out for $d = 0$ spacetime is conformally flat, $ds^2 \sim w^{1/12}(\eta_{ab}dx^a dx^b)$ where $w = B^3$. Note that even now the solution is not asymptotically flat. This following the works of Wiltshire [29] for spherically symmetric scalar black holes is due to the fact that the potential attains its minimum value only at an infinite value for the scalar field ϕ . The only cases where asymptotic flatness or constant curvature is obtained is $\gamma = 0$ and also by considering $\gamma = 1/2$ in the string frame.

For the particular case of $\gamma = 3/2$ i.e. the Type I string we obtain,

$$ds^2 = B^{1/4} \left[-\eta(2\alpha e^{3\phi_0/2} \ln B - d/2) dt^2 + \frac{dB^2}{\eta(2\alpha e^{3\phi_0/2} \ln B - d/2)} + dx_8^2 \right] \quad (21)$$

$$\phi = \phi_0 - \frac{3}{2} \ln B$$

In this case the sign of d does not affect the form of the solution. We will always get a black hole horizon with $\eta = 1$ and a cosmological horizon with $\eta = -1$ situated at $B'_H = 0$. Spacetime is singular at $B = 0$ as the Ricci scalar is given by,

$$R = \pm \frac{1}{8} \frac{-20\alpha + 18\alpha \ln(B) + 9 - d/2}{B^{9/4}}$$

Again for large B the curvature scalar R vanishes signaling the good behavior of the space-time geometry coupled to matter. However the Weyl curvature now turns out to be dependent on α . Since $\alpha \neq 0$ solutions will *not* be conformally flat. Hence again the dilaton potential rolls down to its vacuum value at minus infinity but spacetime is not asymptotically flat or of constant curvature.

IV. GENERAL ONE-DIMENSIONAL SOLUTIONS

The above is only one subset of the static solutions one can find. In this section, we shall extract the general one dimensional static or time-dependent solution to (5). Since we are

looking for 1-dimensional solutions we will either have $\mathcal{A}_u = \mathcal{A}_w$ (for time dependent) or $\mathcal{A}_u = -\mathcal{A}_w$ for static solutions, where \mathcal{A} stands for, B , ν and ϕ . It suffices in view of the form of our metric to find the static solutions, so let us concentrate on this case. The field equations are written,

$$B'' = -2\alpha B^{1/(D-2)} e^{2\nu+\gamma\phi} \quad (22a)$$

$$\nu'' + \frac{1}{4}\phi'^2 = -\frac{\alpha}{D-2} B^{-(D-3)/(D-2)} e^{2\nu+\gamma\phi} \quad (22b)$$

$$\phi'' + \frac{B'}{B}\phi' = 2\alpha\gamma B^{-(D-3)/(D-2)} e^{2\nu+\gamma\phi} \quad (22c)$$

$$2B'\nu' - B'' = \frac{B}{2}\phi'^2 \quad (22d)$$

where $'$ denotes the derivative⁴ with respect to z . Now we make use of similar form in the non-homogeneous terms in the wave equations i.e. using (22a) we can integrate (22c),

$$\phi' = \frac{1}{B}(c - \gamma B') \quad (23)$$

and then in turn, (22a), (22d) in (22b) yield after direct integration,

$$2\nu' = \frac{1}{B}(d + \frac{D-1}{D-2}B') \quad (24)$$

with c and d arbitrary integration constants. From (23) we see that $c = 0$ corresponds to the dilaton solution discussed in the previous section and hence d is again the Weyl parameter. Replacing now the above expressions for ϕ' and ν' in (22d) we get a second order differential equation for B ,

$$(d + \gamma c)\frac{B'}{B} - s\frac{B'^2}{B} - B'' - \frac{c^2}{2B} = 0 \quad (25)$$

where s is given by (13). This equation is consistent modulo a constant with the remaining equation (22a) and this can be shown by differentiating both and comparing. We will make use of this fact later on to fix our constants. All we need to do now is solve (25) and then find ϕ and ν by direct integration in some adequate coordinate system. Equation (25) is autonomous hence we coordinate transform our metric setting $B' = p$ and hence $B'' = \frac{dp}{dB}p$. The above equation then boils down to,

$$p\frac{dp}{dB}B = -sp^2 + (d + \gamma c)p - \frac{c^2}{2} \quad (26)$$

and hence for $s \neq 0$

$$-s\ln B(p) = \int \frac{pdp}{f(p)} \quad (27)$$

⁴The t-dependent equations are obtained by simply replacing α by $-\alpha$ in (22)

where $f(p)$ is a second degree polynomial of p given by,

$$f(p) = p^2 - \frac{d + \gamma c}{s}p + \frac{c^2}{2s} \quad (28)$$

Depending on the type of roots of this polynomial we shall obtain different solutions. The roots of the polynomial will generically correspond to singular points of spacetime. Furthermore noting that $dz = dB/p$ the fields ϕ and ν are directly integrated as functions of p to give,

$$\phi(p) = \ln B^{-\gamma} - \frac{c}{s} \int \frac{dp'}{f(p')} \quad (29a)$$

$$2\nu(p) = \ln B^{(D-1)/(D-2)} - \frac{d}{s} \int \frac{dp'}{f(p')} \quad (29b)$$

Since all our fields are given as functions of p we coordinate transform using (26) whereby $dz^2 = \frac{B^2 dp^2}{s^2 f(p)^2}$ and hence,

$$ds^2 = e^{2\nu} B^{-\frac{D-3}{D-2}} (-dt^2 + \frac{B^2 dp^2}{s^2 f(p)^2}) + B^{\frac{2}{D-2}} dx_8^2 \quad (30)$$

We start by analyzing the roots of the polynomial $f(p)$. Its discriminant is given by,

$$\Delta = \frac{1}{s^2} [(d + \gamma c)^2 - 2c^2 s]$$

Two possibilities arise according to the sign of s . If $s > 0$ then we can have two distinct real roots p_1 and p_2 , one double root and no real roots. For $s < 0$ we *always* have $\Delta \geq 0$. For $D = 10$ we note again the particular role played by $\gamma = 3/2$, mapping s to the origin. Indeed from (26), if $s = 0$ then we have a first degree polynomial. For $\gamma < 3/2$ we will always have 2 real roots. For $\gamma > 3/2$ and hence for the heterotic string, $\gamma = 5/2$, the whole spectrum of possible solutions will be permitted. So depending on the nature of the discriminant Δ we classify our solutions to Class I (two distinct real roots), Class II (double root) and Class III (imaginary roots) solutions. We also have to examine the case $s = 0$ separately.

A. Class I solutions

In this case the two distinct real roots of $f(p)$ are given by, $p_{1,2} = \frac{d+\gamma c}{2s} \pm \frac{p_0}{2}$ with $p_0 = \sqrt{\Delta} > 0$ and we choose $p_2 < p_1$. The Class I fields are obtained directly from (27) and then in turn from (29),

$$B(p) = B_0 \frac{|p - p_2|^{\frac{p_2}{sp_0}}}{|p - p_1|^{\frac{p_1}{sp_0}}}, \quad e^{2\nu} = e^{2\nu_0} B^{\frac{D-1}{D-2}}(p) q^d(p), \quad e^\phi = e^{\phi_0} B^{-\gamma}(p) q^c(p)$$

where B_0 , ν_0 and ϕ_0 are constants of integration and,

$$q(p) = \left| \frac{p - p_2}{p - p_1} \right|^{\frac{1}{sp_0}}$$

The next step is to relate α to our integration constants: noting from (25) that $B''B = -sf(p)$ and replacing the above solutions into (22a) we get

$$f(p)s = 2\eta\alpha e^{2\nu_0+\gamma\phi_0} B_0^{-2s} |f(p)|$$

Here $\eta = 1$ means that p is spacelike and $\eta = -1$ timelike. Therefore for $s > 0$ we will have spacelike solutions for $p > p_1$ or $p < p_2$ whereas timelike solutions will be obtained in the finite interval $p_2 < p < p_1$ whereas for $s < 0$ the situation is interchanged. Once the sign is determined the integration constants are related by,

$$|s| = 2\alpha e^{2\nu_0+\gamma\phi_0} B_0^{-2s} \quad (31)$$

Let us explicitly write and analyze the solution for $D = 10$ and $\gamma = 5/2$ ($s = 2$). All $s > 0$ solutions will have the same behavior. The solutions then are backgrounds to the non-supersymmetric heterotic string.

Now p_1 and p_2 are singular points for the fields and they either stand for curvature singularities or coordinate singularities. For spacelike p such that $p > p_1$ we perform a change of origin, $p - p_1 \rightarrow p$ and use (31) to fix the constants. After some algebra the solution is written,

$$ds^2 = \frac{(p+p_0)^{\frac{p_1-p_0}{8p_0}}}{p^{\frac{p_1}{8p_0}}} \left[- \left(\frac{p+p_0}{p} \right)^{\frac{d}{2p_0}} dt^2 + dx_8^2 + \left(\frac{p+p_0}{p} \right)^{\frac{d}{2p_0} + \frac{p_1}{p_0}} \frac{dp^2}{4\alpha e^{5\phi_0/2} (p+p_0)^3 p^2} \right] \quad (32)$$

$$\phi = \phi_0 + \left(\frac{c}{2p_0} - \frac{5p_1}{4p_0} + \frac{5}{4} \right) \ln(p+p_0) + \left(\frac{5p_1}{4p_0} - \frac{c}{2p_0} \right) \ln p \quad (33)$$

where $p > 0$.⁵ and therefore the $p = -p_0$ singularity is never attained for a positive p . As it stands now Class I solutions depend on two integration parameters c and d . The case $c = 0$ was treated in the previous section where upon making the coordinate transformation $p = \alpha B^{-2} + d/2$ we obtain the dilaton black hole solutions (19). What is noteworthy here is that $p = 0$ is no longer a singular point of spacetime, it is now a horizon screening the singularity at the second root of the polynomial $f(p)$. Indeed the singular part of the dilaton in (33) drops out for $c = 0$. Alternatively we can gauge away the c -dependence modulo the sign. The constant c plays then the role of a topological index characterizing the dilaton field and we shall obtain different solutions for $c = 0, \pm 1$.

So let us now suppose that $c \neq 0$. Then consider $p \rightarrow p_0 p$ relabel d by d/c and set $\eta = |c|/c$,

$$ds^2 = [p(p+1)]^{-1/16} \left(\frac{p+1}{p} \right)^{\frac{\eta(d+5/2)}{16\chi}} \left[- \left(\frac{p+1}{p} \right)^{\frac{\eta d}{\chi}} dt^2 + dx_8^2 + \frac{\left(\frac{p+1}{p} \right)^{\frac{\eta(3d+5/2)}{2\chi}} dp^2}{4\alpha e^{5\phi_0/2} [(p+1)p]^{5/2}} \right] \quad (34)$$

$$\phi = \phi_0 + \frac{5}{8} \ln[p(p+1)] + \frac{5\eta(d+9/10)}{8\chi} \ln \left[\frac{p}{p+1} \right] \quad (35)$$

⁵The spacelike solutions with $p < p_2$ are obtained from (32) by interchanging p by $p + p_0$.

where $\eta = \pm 1$, $\chi = [(d + 9/2)(d + 1/2)]^{1/2}$ and from positivity of the discriminant we have $d \in (-\infty, -9/2) \cup (-1/2, +\infty)$. Notice how (34), (35) are symmetric under $p \leftrightarrow p + 1$ and $\eta = 1 \leftrightarrow \eta = -1$. We can portray (34) in the string frame via the conformal rescaling $g^{(S)} = e^{\phi/2}g$ and it turns out that they share the same features as the Einstein solutions (34).

We now determine the nature of the singularity for $p = 0$ in (34). A direct calculation of the Ricci scalar gives,

$$R \sim [p(p + 1)]^{9/16} \left[\frac{p}{p + 1} \right]^{\frac{25\eta(d+9/10)}{16\chi}} \quad (36)$$

and hence R blows up at $p = 0$ if and only if $\eta(d + 9/10) < 0$ (calculation of $R_{ab}R^{ab}$ and $R_{abcd}R^{abcd}$ yields the same result). Note from (35) that the dilaton potential exhibits the same behavior and furthermore the Ricci scalar is always singular for large p . The behavior of the curvature tensor is related to the topology of the solution. Indeed consider the proper distance in the p direction defined as,

$$\int_0^\infty dp \sqrt{g_{pp}}$$

It is easy to see from (34) that proper distance is finite if and only if, $\eta(d + 9/10) < 0$ i.e. given the interval $d \in (-\infty, -9/2) \cup (-1/2, +\infty)$ we have that $d < -9/2$ for $\eta = 1$, and $d > -1/2$ for $\eta = -1$ yield a compact p direction. Hence we deduce that the p -direction is compact and our solution has the topology of an interval times a 9-dimensional manifold if and only if the Ricci tensor blows up at $p = 0$. Hence compactness in the p -direction yields a singular behavior of spacetime with two naked singularities at $p = 0$ and at $p \sim \infty$ (the singularity at infinity is at a finite proper distance). Furthermore the ‘compact’ solutions (34) solutions are not asymptotically flat. For large p spacetime takes the approximate form,

$$ds^2 \sim p^{-1/8} [\eta_{\mu\nu} dx^\mu dx^\nu + p^{-5} \frac{dp^2}{4\alpha e^{5\phi_0/2}}]$$

and the dilaton

$$\phi \sim \phi_0 + \frac{5}{4} \ln p$$

which is just the $SO(9)$ solution (20).

Alternatively if proper distance is infinite then $p = 0$ is a coordinate singularity. It turns out that this solution (unlike the case $c = 0$) cannot be extended in the timelike region so we make a coordinate transformation making $p = 0$ spatial infinity, $p = 1/u$. Then again we find that the solution is not asymptotically flat. From (35) we get,

$$\phi = \phi_0 + \frac{5}{8} \ln \frac{1+u}{u^2} + \frac{5\eta(d+9/10)}{8\chi} \ln \left[\frac{1}{1+u} \right]$$

and we see that the dilaton approaches minus infinity for large u rolling down the potential well given that $\eta(d + 9/10) > 0$ (see fig 1).

Let us now turn to the time-dependent solutions. As we mentioned above in this case we have $f(p) < 0$ and therefore $p_2 < p < p_1$ varies in a finite interval. Choosing p_2 as our

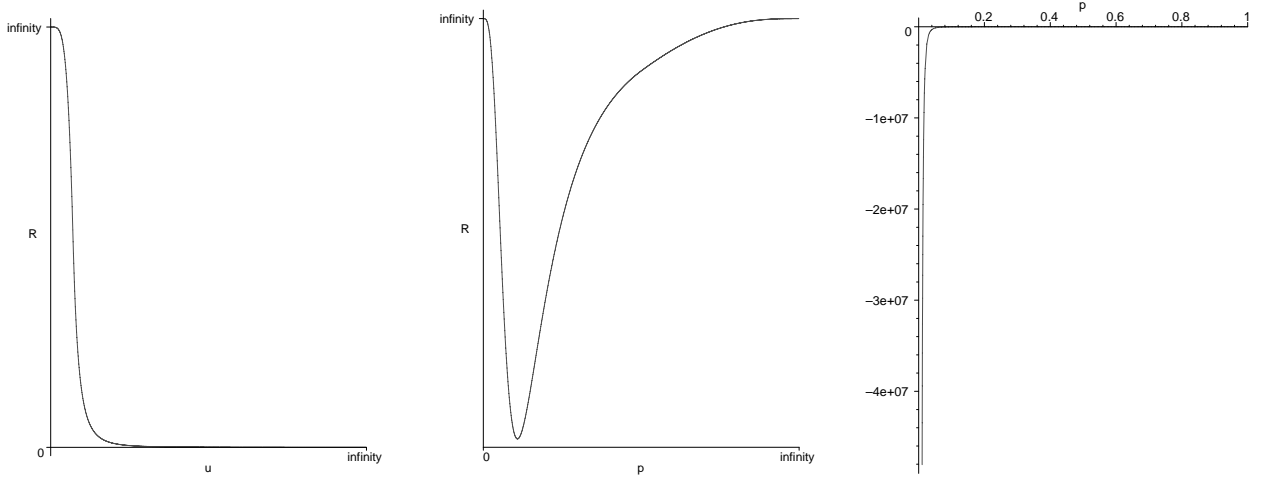


FIG. 1. Plot of ricci scalar for $d = -5$ for spacelike Class I $\eta = -1, +1$ and timelike Class I with $\eta = -1$ respectively

origin we note that $dt^2 = \frac{B^2 dp^2}{s^2 f^2(p)}$. The time dependent solution is obtained by considering solution (32) and replacing, $p + p_0 \rightarrow p$ and $-dt^2 \rightarrow dz^2$. Proceeding as for the spacelike case the timelike solutions for $c \neq 0$ simplify to,

$$ds^2 = [p(1-p)]^{-1/16} \left(\frac{p}{1-p} \right)^{\frac{\eta(d+5/2)}{16\chi}} \left[-\frac{\left(\frac{p}{1-p} \right)^{\frac{\eta(3d+5/2)}{2\chi}} dp^2}{4\alpha e^{5\phi_0/2} [(1-p)p]^{5/2}} + \left(\frac{p}{1-p} \right)^{\frac{\eta d}{\chi}} dz^2 + dx_8^2 \right] \quad (37)$$

$$\phi = \phi_0 + \frac{5}{8} \ln[p(1-p)] + \frac{5\eta(d+9/10)}{8\chi} \ln \left[\frac{1-p}{p} \right] \quad (38)$$

where $0 < p < 1$. The Ricci scalar, obtained by (36) with $p+1 \rightarrow p$ is singular at one of the two extremities of the interval $p = 0$ or $p = 1$. Indeed for $\eta = -1$ say, $p = 1$ is a coordinate singularity and $p = 0$ is a naked singularity. Changing the sign of η does not produce a new solution but merely changes the direction of time. The direction of time is determined by demanding that the scalar field tends asymptotically to minus infinity. We should also note that proper time is always infinite although our coordinate interval for p is finite.

The form of the metric (34) dictates that for $d = 0$ i.e. for vanishing Weyl tensor we will get the general solution with $SO(9)$ symmetry.

d=0: Under this assumption we have, two distinct cases depending on the sign of η . For $\eta = -1$ we will have finite proper distance in the p direction and two naked singularities at $p = 0$ and at coordinate infinity whereas for $\eta = 1$ the proper distance is infinite with $p = 0$ merely a coordinate singularity. Explicitly for $\eta = -1$ we obtain from (34) and (35) the solution,

$$ds^2 = (p+1)^{-1/6} p^{1/24} \eta_{\mu\nu} dx^\mu dx^\nu + (p+1)^{-7/2} p^{-13/8} dp^2 (4\alpha)^{-1} e^{-5\phi_0/2}$$

$$\phi = \phi_0 + \ln p^{1/4} + \ln(p+1)$$

This is the $SO(16) \times SO(16)$ [15] solution as can be seen by considering $p = ch^2(\sqrt{\alpha}y)$, where it was noted that the solution has an effective 9-dimensional behavior with the 9 dimensional Planck and Yang-Mills couplings finite.

The $\eta = 1$ can be obtained explicitly by replacing p by $p + 1$ and vice-versa. In this case spacetime is well behaved at $p = 0$ the point in question portraying a coordinate singularity and hence as before we take $u = 1/p$,

$$ds^2 = u^{1/8}(u+1)^{1/24}\eta_{\mu\nu}dx^\mu dx^\nu + (u+1)^{-13/8}u^{9/8}dp^2(4\alpha)^{-1}e^{-5\phi_0/2}$$

$$\phi = \phi_0 - 5/4\ln u + \ln(u+1)$$

We now have a naked singularity at $u = 0$ and dilaton matter tends to 0 for u large with ϕ going to minus infinity.

The unique timelike solution in the Einstein frame is,

$$ds^2 = -(1-p)^{-7/2}p^{-13/8}\frac{dp^2}{4\alpha e^{5\phi_0/2}} + (1-p)^{-1/6}p^{1/24}\eta_{\mu\nu}dx^\mu dx^\nu$$

$$\phi = \phi_0 + \ln(1-p) + \frac{1}{4}\ln p$$

with $0 < p < 1$ where η is gauged away since it only interchanges $p \rightarrow 1-p$. In the case of $SO(9)$ symmetry we have a homogeneous and isotropic perfect fluid generated by the dilaton field. The energy density and pressure give,

$$\rho(p) = \frac{1}{16}\alpha e^{5\phi_0/2}\frac{(3p+1)^2(1-p)^{5/2}}{p^{3/8}}$$

$$P(p) = \frac{1}{16}\alpha e^{5\phi_0/2}(41p^2 - 26p + 1)(1-p)^{3/2}p^{-3/8}$$

Note then how the fluid is concentrated in the region of the initial cosmological singularity (figure 2). We see that pressure changes sign accordingly to the scalar curvature of spacetime (figure 2). This behavior is reminiscent of cosmological models with scalar fields [32] in 4 dimensional standard cosmology and in extension to quintessence models which strive into explaining the present acceleration [33] of our Universe. Note that as $p \rightarrow 1$ the dilaton field approaches minus infinity and therefore rolls down the potential well with dilatonic matter dissipating in that region. However spacetime is not asymptotically flat.

The last solution of $SO(9)$ symmetry is simply obtained by imposing $c = d = 0$ (20). Strictly speaking it belongs to Class II solutions but we include it here for completeness. This solution exhibits the asymptotic behavior of all Class I solutions of compact proper distance. It reads,

$$ds^2 = p^{-1/8}\left(\eta_{\mu\nu}dx^\mu dx^\nu + \frac{dp^2}{4\alpha e^{5\phi_0/2}p^5}\right)$$

$$\phi = \phi_0 + 5/4\ln p$$

These three solutions are the unique $SO(9)$ solutions in the non-supersymmetric heterotic theory. The important point is that these solutions are not asymptotically, flat or constant curvature, even though the scalar field rolls down a global minimum.

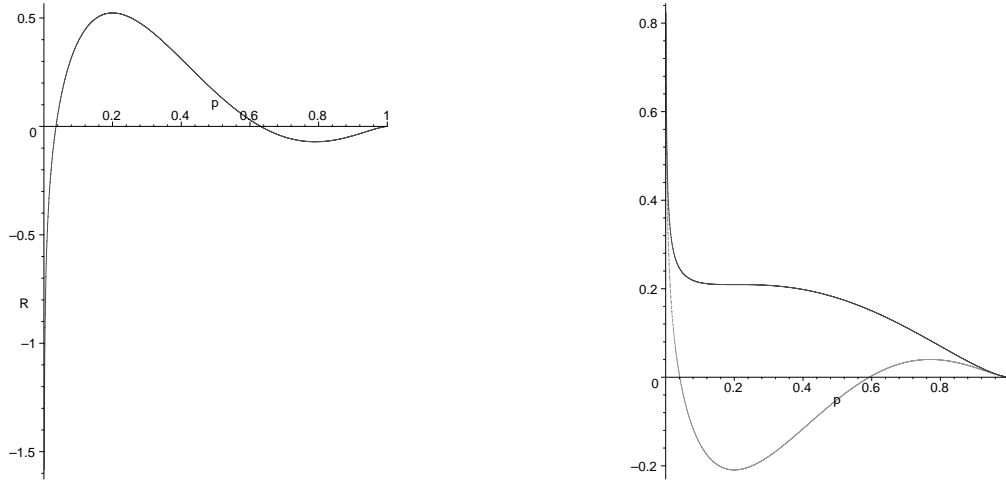


FIG. 2. Plot of the scalar Ricci and ρ , P for the $SO(9)$ time-dependent solution. We have set $\phi_0 = 0$, $\alpha=1$.

B. Class II solutions

This case is defined by $\Delta = 0$ which implies that $d = c(-\gamma \pm \sqrt{2s})$, $s > 0$ and the double root of $f(p)$ is given by $p_1 = \pm \frac{c}{\sqrt{2s}}$. Then we use (25) and (29) and performing a change of origin,

$$ds^2 = p^{\frac{-2}{(D-2)s}} e^{\frac{2p_1}{(D-2)ps}} \left[e^{\frac{d}{sp}} \left(-dt^2 + \frac{e^{\frac{2p_1}{ps}} dp^2}{e^{\gamma\phi_0} 2\alpha s p^{4+\frac{2}{s}}} \right) + dx_8^2 \right] \quad (39)$$

$$\phi = \phi_0 + \frac{\gamma}{s} \ln p + \frac{c - \gamma p_1}{sp} \quad (40)$$

for $p > 0$.

Consider now the 10 dimensional heterotic case where $\gamma = 5/2$ i.e. $s = 2$. We have two static solutions characterized by the roots, $p_1 = \pm c/2$ and $d = -c/2$, $d = -9c/2$ respectively. Consider first the couple $p_1 = c/2$, $d = -c/2$. By setting $p \rightarrow |c|p/4$ the solution is given by,

$$ds^2 = p^{-1/8} e^{\frac{\eta}{4p}} \left(-e^{\frac{-\eta}{p}} dt^2 + p^{-5} e^{\frac{\eta}{p}} \frac{dp^2}{e^{5\phi_0/2} 4\alpha} + dx_8^2 \right) \quad (41)$$

$$\phi = \phi_0 + \frac{5}{4} \ln p - \frac{\eta}{2p} \quad (42)$$

where as before $\eta = \pm 1$. Not surprisingly for large p (41) asymptotes the $c = d = 0$ solution (20). In the string frame the solution is,

$$ds^2 = p^{1/2} \left(-e^{\frac{-\eta}{p}} dt^2 + p^{-5} e^{\frac{\eta}{p}} \frac{dp^2}{e^{5\phi_0/2} 4\alpha} + dx_8^2 \right)$$

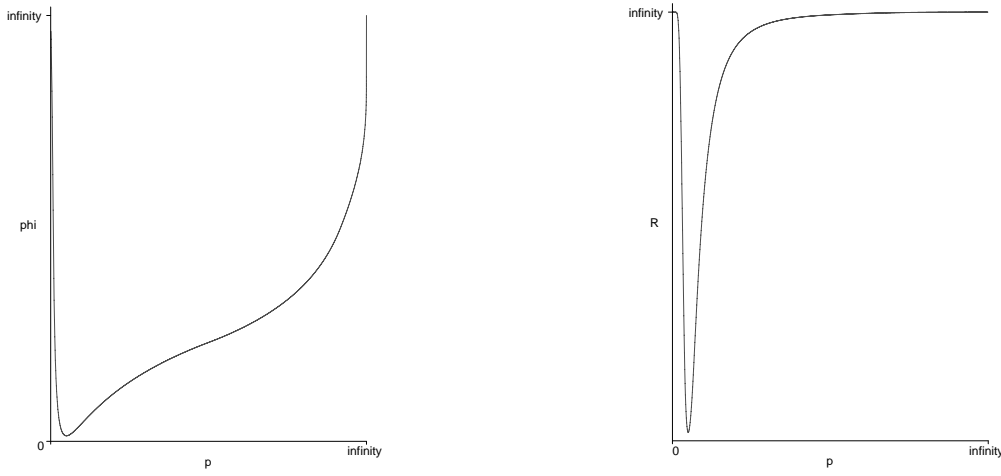


FIG. 3. Plot of the dilaton field and the Ricci scalar curvature for $\eta = -1$

As for Class I solutions (41) has a totally different behavior depending on the sign of η . Indeed for $\eta = -1$ the proper distance in the p direction is finite. Hence spacetime is spontaneously compactified in the p direction both in the Einstein and in the string frame. Furthermore calculation of the Ricci scalar shows that for $\eta = -1$ (41) has two naked singularities at $p = 0$ and at coordinate infinity,

$$R = \frac{\alpha e^{\frac{5\phi_0}{2}}}{8} p^{\frac{9}{8}} e^{-\frac{5\eta}{4p}} (45p^2 + 20\eta p + 4)$$

The dilaton potential (and the Ricci scalar) have a global minimum at finite non-zero p where the exponential term takes over the power law (see figure 3).

However for $\eta = 1$ proper distance is infinite in p and accordingly $p = 0$ is only a coordinate singularity with spacetime diverging at spatial infinity. We coordinate transform (41) bringing $p = 0$ to infinity, $u = 1/p$, and (41) gives,

$$ds^2 = u^{1/8} e^{\frac{u}{4}} (-e^{-u} dt^2 + u e^u \frac{du^2}{e^{5\phi_0/2} 4\alpha} + dx_8^2)$$

$$\phi = \phi_0 + \frac{5}{4} \ln(1/u) - \frac{u}{2}$$

Hence now we have a naked singularity at $u = 0$ and at large u dilaton matter tends to 0. Note in the figure 4 how the absolute minimum of ϕ at minus infinity is attained as $u \rightarrow \infty$ with the curvature scalar going to 0.

When $p_1 = -c/2$ and $d = -9c/2$ we get in the same fashion as before,

$$ds^2 = p^{-1/8} e^{\frac{-\eta}{4p}} (-e^{\frac{-9\eta}{p}} dt^2 + p^{-5} e^{\frac{-11\eta}{p}} dp^2 e^{-5\phi_0/2} (4\alpha)^{-1} + dx_8^2)$$

$$\phi = \phi_0 + \frac{5}{4} \ln p + \frac{9\eta}{2p}$$

Again for $\eta = 1$ the p -direction is compact with the according behavior of the curvature tensor whereas $\eta = -1$ yields infinite proper length.

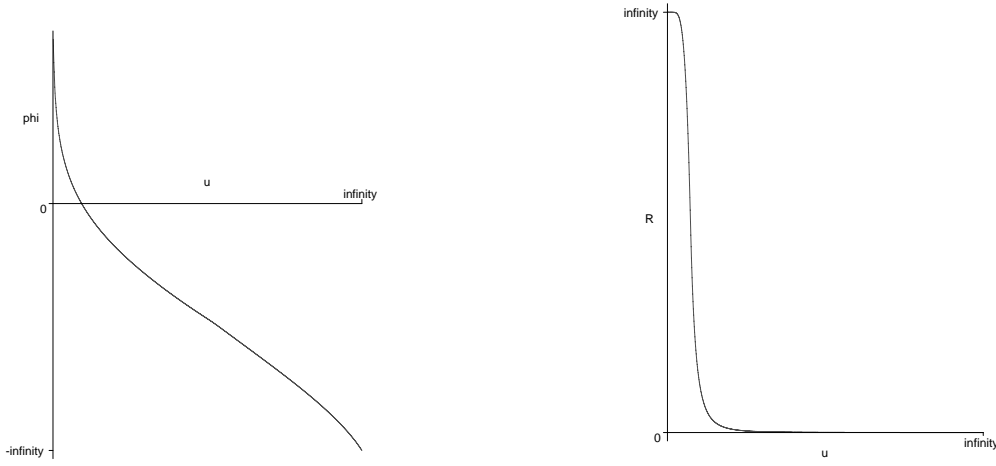


FIG. 4. Plot of the dilaton field and its potential for $\eta = 1$

We note again the similar topological role played by the integration constant c and the persisting property, proper distance is compact iff curvature diverges at the endpoints of the proper interval.

C. Class III solutions

In this case we have that $f(p) > 0$ for all p . We can therefore anticipate that there will not be any singular points for finite values of coordinate p which can therefore vary on the entire real line. This class of solutions is only attained for $s > 0$ and the negative discriminant is given by $-\Delta = \frac{c^2\psi^2}{s^2}$ where we have set $\psi^2 = 2s - (d/c + \gamma)^2$ a positive real number. We proceed as before using (25) and then (29) to find B , ν and ϕ . Performing a change of origin, $p - \frac{d+\gamma c}{2s} \rightarrow p$ to simplify notation we find the D -dimensional Class III metric,

$$ds^2 = \left[p^2 + \frac{\psi^2}{4s^2}\right]^{-\frac{1}{(D-2)s}} e^{-\frac{2(d+\gamma c)}{s\psi(D-2)}q(p)} \left[e^{-\frac{2d}{\psi}q(p)} \left(-dt^2 + \frac{e^{-\frac{2(d+\gamma c)}{s\psi}q(p)} dp^2}{2\alpha s e^{5\phi_0/2} [p^2 + \frac{\psi^2}{4s^2}]^{4+\frac{1}{s}}} \right) + dx_8^2 \right] \quad (43)$$

$$\phi = \phi_0 + \frac{\gamma}{2s} \ln \left[p^2 + \frac{\psi^2}{4s^2} \right] + \frac{\gamma(d + \gamma c) - 2cs}{s\psi} q(p)$$

where $p \in]-\infty, \infty[$ and we have set $q(p) = \text{Arctan}(\frac{2ps}{\psi})$.

We now examine the $D = 10$, heterotic $\gamma = 5/2$ background, whereby we set $p \rightarrow p\psi/4$, we relabel d/c by d and therefore $-9/2 < d < -1/2$ for positive ψ . We now perform the coordinate transformation $\tan q = 4p/\psi$ bringing spatial infinity to a finite value with $q \in]-\pi/2, \pi/2[$. After some algebra and using (31) we obtain

$$ds^2 = (\cos q)^{1/8} e^{-\frac{d+5/2}{8\psi}q} \left(-e^{-\frac{2dq}{\psi}} dt^2 + \frac{\cos q e^{-\frac{3d+5}{\psi}q}}{4\alpha e^{5\phi_0/2}} dq^2 + dx_8^2 \right)$$

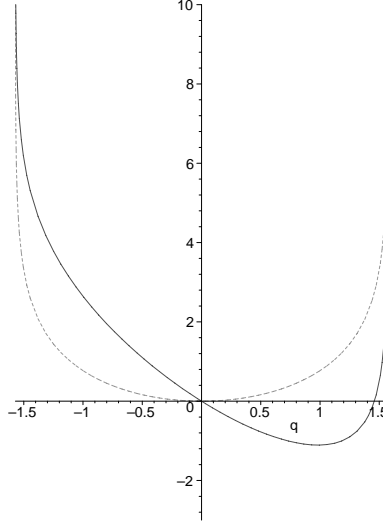


FIG. 5. Plot of the dilaton field for $d = -9/10$ (dotted line) and $d = -7/2$

$$\phi(q) = \phi_0 - \frac{5}{4} \ln(\cos q) + 5 \frac{d + 9/10}{4\psi} q$$

Now it is obvious that given the finite interval at which our spacelike coordinate q varies that the proper distance in the q -direction will always be finite. Hence Class III solutions are always compact in the q direction. Furthermore from the dilaton field we see that spacetime curvature diverges as q approaches $\pm\pi/2$. In the string frame the Class III solution reads,

$$ds^2 = (\cos q)^{-1/2} e^{\frac{2d+1}{4\psi} q} \left(-e^{-\frac{2dq}{\psi}} dt^2 + \frac{e^{-\frac{3d+5}{\psi} q}}{4\alpha e^{5\phi_0/2}} \cos q dq^2 + dx_8^2 \right)$$

and the above characteristics remain true. The dilaton potential has a unique local minimum for positive q and diverges at the ends of the interval. The Ricci scalar is well behaved up until we reach $q = \pm\pi/2$ where it explodes exponentially fast. Note the particular value $d = -9/10$ which was the value that separated compact from infinite proper distance for the class I solutions. Here it yields an even function for the dilaton. For $-9/2 < d < -9/10$ we have a local minimum which approaches $\pi/2$ as $d \rightarrow -9/2$ and conversely for $-9/10 < d < -1/2$ (see figure 5).

V. THE CRITICAL CASE AND THE OPEN STRING

When $s = 0$ the critical value of γ is given (13) by $\gamma = \sqrt{\frac{2(D-1)}{(D-2)}}$ and one has to start from (26) to obtain

$$\ln B = (d + \gamma c)^{-1} \int \frac{p dp}{f(p)}$$

where $f(p)$ is now a first degree polynomial,

$$f(p) = p - \frac{c^2}{2(d + \gamma c)} \quad (44)$$

We label the unique root by, $p_1 = \frac{c^2}{2(d + \gamma c)}$ and we anticipate a spacetime singularity at this point. The components are easily integrated from (26), (23) and (24) and yield,

$$B(p) = B_0 e^{\frac{p}{d+3c/2}} |p-p_1|^{\frac{c^2}{2(d+3c/2)^2}}, \quad e^{2\nu} = e^{2\nu_0} B^{(D-1)/(D-2)} |p-p_1|^{\frac{d}{d+3c/2}}, \quad e^\phi = e^{\phi_0} B^{-3/2} |p-p_1|^{\frac{c}{d+3c/2}}$$

Using (25) and (22a) we see that the integration constants are related by,

$$(d + \gamma c)f(p) = -2\alpha e^{2\nu_0 + \gamma\phi_0} |f(p)| \quad (45)$$

which states in particular that $d + \gamma c < 0$ for static solutions and $d + \gamma c > 0$ for time-dependent solutions⁶. Let us start with the former case. Performing a change of origin $p - p_1 \rightarrow p$ and considering $p > 0$ we obtain the following static solution,

$$ds^2 = -e^{\frac{2p}{(D-2)(d+\gamma c)}} p^{\frac{2p_1}{(D-2)(d+\gamma c)}} \left[p^{\frac{d}{d+\gamma c}} \left(-dt^2 + \frac{e^{\frac{2p}{d+\gamma c}} p^{\frac{2p_1}{d+\gamma c} - 2} dp^2}{(d + \gamma c)^2 e^{\gamma\phi_0}} \right) + dx_8^2 \right] \quad (46)$$

$$\phi = \phi_0 - \frac{\gamma}{d + \gamma c} p + \frac{c - \gamma p_1}{d + \gamma c} \ln p \quad (47)$$

Let us now examine the case $D = 10$, $\gamma = 3/2$ which corresponds to $SO(8)$ backgrounds for the Type I string. As before we can absorb the constant c modulo a sign by setting $p \rightarrow p|c|$, putting $\eta = |c|/c$ and relabeling $d \rightarrow d/c$. The case of $c = 0$ was examined in section III. For $p > 0$ we have from (45) that $d + 3/2 < 0$ for $\eta = 1$ (or $d + 3/2 > 0$ for $\eta = -1$) for a spacelike solution whereas we have $d + 3/2 < 0$ and $\eta = -1$ (or $d + 3/2 > 0$ for $\eta = 1$) for timelike solutions.

The $USp(32)$ spacelike solution ($c \neq 0$) written in the Einstein frame is,

$$ds^2 = e^{\frac{\eta p}{4(d+3/2)}} p^{\frac{1}{8(d+3/2)^2}} \left[p^{\frac{d}{d+3/2}} \left(-dt^2 + \frac{e^{\frac{2p\eta}{d+3/2}} p^{\frac{1}{(d+3/2)^2} - 2} dp^2}{(d + 3/2)(-\eta)e^{3\phi_0/2} 2\alpha} \right) + dx_8^2 \right] \quad (48)$$

$$\phi = \phi_0 - \frac{3\eta}{2d + 3} p + \frac{d + 3/4}{(d + 3/2)^2} \ln p$$

Quite generically the behavior of solutions (48) near $p = 0$ is controlled by the powers of p whereas the asymptotic large p behavior is controlled by the exponentials. A simple calculation shows that the spacelike solutions are always compact in the p direction. Then as before we have two naked singularities, one at $p = 0$ and one at finite proper distance (but at coordinate infinity). So once more we see that proper distance is compact iff the curvature diverges at the endpoints of the p -interval. To illustrate the above results let us consider the explicit example where $d = -5/2$. The curvature scalar reads,

$$R = \frac{\eta}{16} e^{3\phi_0/2} \alpha e^{9p\eta/4} \frac{36p^2 - 44\eta p + 49}{p^{29/8}}$$



FIG. 6. Plot of the Ricci scalar and dilaton field for $d = -5/2$, $\eta = 1$ (static case).

Now notice when $\eta = 1$ (spacelike case) curvature diverges at both ends with the dilaton field acquiring a global minimum at the root of the polynomial for finite p . In the string frame $g_s = e^{\phi/2}g$ all the properties discussed go through as in the Einstein frame solution (48).

The timelike solution reads,

$$ds^2 = e^{\frac{\eta p}{4(d+3/2)}} p^{\frac{1}{8(d+3/2)^2}} \left[p^{\frac{d}{d+3/2}} \left(-\frac{e^{\frac{2p\eta}{d+3/2}} p^{\frac{1}{(d+3/2)^2}-2} dp^2}{\eta(d+3/2)e^{3\phi_0/2}2\alpha} + dz^2 \right) + dx_8^2 \right]$$

where p is a positive timelike coordinate. The fact that $d + 3/2$ is now positive changes the asymptotic behavior of the solutions. Indeed proper time is always infinite iff $\eta = -1$ and now dilaton matter is concentrated around $p = 0$ and is exponentially damped for larger p . Accordingly the dilaton rolls down the potential well with $\phi \rightarrow -\infty$ as $p \rightarrow +\infty$, however, spacetime is not asymptotically flat. Weyl curvature (depending on parameter d) takes over at large p inducing purely gravitational forces. Although the dilaton rolls down the potential well, spacetime is not trivial as one could have naively expected.

It is now instructive to look at some asymptotic values of c and d . The first case we consider is well apart, $d = -3/2$, since for this value the polynomial $f(p) = -c^2/2$ is just a non-zero constant.

d=-3/2: The solution reads,

$$ds^2 = e^{-p^2/4+3p} \left(-dt^2 + e^{-2p^2} \frac{dp^2}{e^{3\phi_0/2}4\alpha} \right) + e^{-p^2/4} dx_8^2$$

$$\phi = \phi_0 + 3p^2/2 - 2p$$

with $p \in R$. Proper distance is again finite and this solution explodes at the tips of the proper interval.

⁶Note that $c = d = 0$ implies from (45) that $\alpha = 0$ and is thus excluded



FIG. 7. Plot of the Ricci scalar and dilaton field for $d = -5/2$ and $\eta = -1$ (timelike case)

d=0: This case gives $SO(9)$ solutions of vanishing Weyl curvature,

$$ds^2 = e^{\frac{p\eta}{6}} p^{\frac{1}{18}} (-dt^2 + dx_8^2) + \frac{e^{-3\phi_0/2}}{3\alpha} e^{\frac{3p\eta}{2}} p^{\frac{-3}{2}} dp^2$$

$$\phi = \phi_0 - \eta p + \frac{1}{3} \ln p$$

For $\eta = -1$ we get the unique spacelike solution of compact proper distance in p and upon making the coordinate transformation $p = \frac{3}{4}\alpha y^2$ and relabeling the constants we recognize the [15] solution with $SO(9)$ symmetry. For $\eta = 1$ we obtain a timelike solution,

$$ds^2 = e^{\frac{p}{6}} p^{\frac{1}{18}} (dx_9^2) - \frac{e^{-3\phi_0/2}}{3\alpha} e^{\frac{3p}{2}} p^{\frac{-3}{2}} dp^2$$

$$\phi = \phi_0 - p + \frac{1}{3} \ln p$$

This solution (see figure 8) is reminiscent of scalar field cosmologies studied in [32]. As time tends to infinity the dilaton field rolls down the potential hill obtaining its vacuum expectation value. The energy density and pressure of the homogeneous and isotropic dilaton fluid read,

$$\rho(p) = \alpha e^{3\phi_0/2} \left(\frac{3}{4}p^2 + \frac{1}{2}p + \frac{1}{12} \right) e^{-3p/2} p^{-1/2}$$

$$P(p) = \alpha e^{3\phi_0/2} \left(\frac{3}{4}p^2 - \frac{1}{2}p + \frac{1}{12} \right) e^{-3p/2} p^{-1/2}$$

and fall exponentially fast to a constant value for large time p . Accordingly the Ricci scalar has a naked singularity at $p = 0$ and for large p tends to a finite value. Again we do not have asymptotic flatness however.

To summarize we saw that the static type I string solutions of $SO(8)$ or indeed of higher symmetry are always of compact proper distance in p and are hence singular at the endpoints of the interval. The timelike solutions exhibit a rather interesting behavior, namely matter is heavily concentrated around the origin and is exponentially dumped as we move away with the dilaton field rolling down the potential hill. However spacetime is not asymptotically flat.

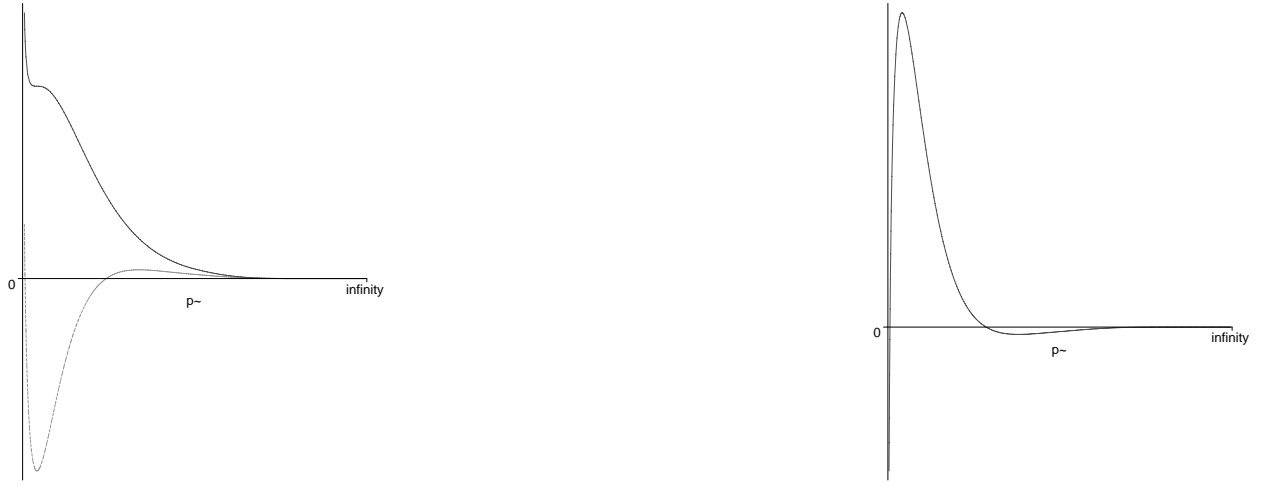


FIG. 8. Plot of the Ricci scalar and energy density, pressure for the timelike $d = 0$ case ($\alpha = 1$, $\phi_0 = 0$) (timelike case)

VI. TREATING THE GENERAL PROBLEM

Let us now go back to the general problem of solving (9) under the light of the one-dimensional solutions. The question we are tackling in this section is the existence of 2-dimensional solutions of (9). Put in a different way we are asking if the general one-dimensional solutions we obtained are the unique solutions of (9)⁷. If this were the case we would obtain an extension of Birkhoff's theorem, in the sense that there would exist a local timelike or spacelike extra Killing vector for our two dimensional system making thus solutions 1-dimensional.

Let us start by analyzing the field equations. We start by noting that using (9a) in (9c) we obtain the *linear-like equation*,

$$B\phi_{uv} + \frac{1}{2}(B_u\phi_v + B_v\phi_u) + \gamma B_{uv} = 0 \quad (49)$$

We hence set without loss of generality,

$$\phi = \tilde{\phi} - \gamma \ln B \quad (50)$$

and (49) simplifies to,

$$(B\tilde{\phi}_u)_v = -(B\tilde{\phi}_v)_u \quad (51)$$

where $\tilde{\phi}$ is now the unknown field. Furthermore let us define,

$$2\chi = 2\nu + \gamma\tilde{\phi} - \frac{\gamma^2}{2}\ln B \quad (52)$$

⁷Discussions with David Langlois and Maria Rodriguez-Martinez have considerably improved our understanding of this section

upon which the system (9) simplifies to,

$$B_{uv} = 2\alpha B^{1/(D-2)-\gamma^2/2} e^{2\chi} \quad (53a)$$

$$(2\chi)_{uv} = \alpha \left(\frac{2}{D-2} - \gamma^2 \right) B^{-\frac{D-3}{D-2}-\gamma^2/2} e^{2\chi} - \frac{1}{2} \tilde{\phi}_u \tilde{\phi}_v \quad (53b)$$

$$\tilde{\phi}_{uv} = -\frac{1}{2B} (\tilde{\phi}_u B_v + \tilde{\phi}_v B_u) \quad (53c)$$

$$(2\chi)_u - [ln(B_u)]_u = \frac{B}{2B_u} \tilde{\phi}_u^2 \quad (53d)$$

$$(2\chi)_v - [ln(B_v)]_v = \frac{B}{2B_v} \tilde{\phi}_v^2 \quad (53e)$$

The independent variables are now B , $\tilde{\phi}$ and χ . Note that α , γ and D appear only in the wave equations (53a) and (53b). Furthermore since (53b) results from the other field equations, (53a) is the only equation depending on the parameters of the problem. Note that as long as $\alpha \neq 0$ we have $B_{uv} \neq 0$ as can be seen from (53a). By the redefinition of the field components we have effectively absorbed the dilaton potential in our new variables. The system (53) corresponds to an energy-momentum tensor consisting of a scalar field with cosmological constant α .⁸

Now from (53a) we can read off,

$$e^{2\chi} = \frac{1}{2\alpha} B_{uv} B^{\gamma^2/2-1/(D-2)} \quad (54)$$

and then replace it in the integrability conditions (53d) and (53e),

$$\tilde{\phi}_u^2 = 2 \frac{B_u B_{uvu}}{B B_{uv}} + 2 \left(\frac{\gamma^2}{2} - \frac{1}{D-2} \right) \frac{B_u^2}{B^2} - 2 \frac{B_{uu}}{B} \quad (55)$$

$$\tilde{\phi}_v^2 = 2 \frac{B_v B_{vvu}}{B B_{uv}} + 2 \left(\frac{\gamma^2}{2} - \frac{1}{D-2} \right) \frac{B_v^2}{B^2} - 2 \frac{B_{vv}}{B} \quad (56)$$

Now in principle we have two unknown functions and four equations to solve. Which of these equations are independent? Suppose we make use only of (55) and (56); can we obtain (53b) and (53c)? Differentiating (55) and (56) with respect to v and u respectively we obtain,

$$2B \tilde{\phi}_{uv} + B_v \tilde{\phi}_u - 2 \frac{B_u}{\tilde{\phi}_u} \mathcal{A} = 0$$

$$2B \tilde{\phi}_{uv} + B_u \tilde{\phi}_v - 2 \frac{B_v}{\tilde{\phi}_v} \mathcal{A} = 0$$

which are symmetric under $u \rightarrow v$. The functional,

$$\mathcal{A} = (ln B_{uv})_{uv} + \left(\frac{\gamma^2}{2} - \frac{1}{D-2} \right) \left(2 \frac{B_{uv}}{B} - \frac{B_u B_v}{B^2} \right)$$

⁸For $\gamma = 1/6$ and $D = 10$ we get from $1/(D-2) - \gamma^2/2 = 1/(n-2)$ an $n = 11$ -dimensional spacetime as one would expect from KK type compactification

is symmetric in u and v . Combining these two equations we get,

$$\left(1 - \frac{B_v \tilde{\phi}_u}{B_u \tilde{\phi}_v}\right) (2B \tilde{\phi}_{uv} + B_u \tilde{\phi}_v + B_v \tilde{\phi}_u) = 0$$

and

$$\left(1 - \frac{B_v \tilde{\phi}_u}{B_u \tilde{\phi}_v}\right) \left((2\chi)_{uv} + \alpha \left(\frac{2}{D-2} - \gamma^2\right) B^{-\frac{D-3}{D-2} - \gamma^2/2} e^{2\chi} - \frac{1}{2} \tilde{\phi}_u \tilde{\phi}_v\right) = 0$$

We recognize as factors (53c) and (53b) which therefore result from (55) and (56) as long as we *don't* have,

$$B_v \tilde{\phi}_u = B_u \tilde{\phi}_v \quad (57)$$

Now to what extend is (57) a relevant equation? Using (53c) and (57) results to $B = B(U(u) + V(v))$ which leads to a one dimensional solution studied in the previous section. Indeed we remind the reader that any function of $U + V$ can be reduced to a z -dependent function using coordinate transformations (17). This is due to the fact that we have imposed $SO(D-2)$ symmetry hence the two remaining dimensions admit 2-dimensional conformal invariance. Therefore any strictly two-dimensional solutions will not obey (57) and hence we deduce that (55), (56) yield the wave equations (53b) and (53c). Conversely the argument follows through in exactly the same way. Hence we deduce the following: Equations (55), (56) are equivalent to equations (53b) and (53c) for *two-dimensional solutions*.

Hence given B we can evaluate χ from (54). Then from (55), (56) evaluate $\tilde{\phi}_u$ and $\tilde{\phi}_v$. Two conditions have then to be met in order to obtain a two-dimensional solution. First of all we must not have (57) for then the solution can be coordinate transformed to a 1-dimensional solution. Secondly $\tilde{\phi}_u$ and $\tilde{\phi}_v$ have to be differentiated with respect to v and u respectively and must yield the same result (equivalently ϕ is a 0-form).

So finding two-dimensional solutions seems a difficult task. However there is a way which gets around this difficulty. Indeed note that integrability conditions (55) and (56) involve the *square* of the derivatives of ϕ in u and v . Hence a way to circumvent (57) is to take ϕ_u and ϕ_v of opposite sign.

Indeed let us apply the above algorithm with $B = e^{A(U(u)+V(v))}$. From (54) we obtain the χ field,

$$e^{2\chi} = \frac{1}{2\alpha} (A'' + A'^2) U' V' B^{\gamma^2/2 + \frac{D-3}{D-2}}$$

and from (55),

$$\tilde{\phi}_u^2 = \frac{2U'^2}{A'' + A'^2} [A' A''' + (2+s) A'' A'^2 + (s+1) A'^4 - A''^2] \quad (58)$$

((56) yields the analogous equation for ' v ') and *prime* stands for the derivative with respect to the argument of the function in question. A simple example to consider is $A = U + V$, [18]. Then it is straightforward to get,

$$e^{2\chi} = \frac{1}{2\alpha} U' V' e^{(\gamma^2/2 + \frac{D-3}{D-2})(U+V)}$$

and

$$\tilde{\phi} = \pm \sqrt{\gamma^2 - 2/(D-2)}(U - V)$$

Note then the $U - V$ dependence of $\tilde{\phi}$ and consequently that (57) is not true and that the solutions are two dimensional. Had we taken the same sign for the derivatives of $\tilde{\phi}$ we would have got a static solution. This sign ambiguity in (55) and (56) breaks the unicity argument of Birkhoff's theorem. Obviously if $\tilde{\phi}$ were a constant this sign ambiguity would not have been possible.⁹

In the generic case of arbitrary A other 2-dimensional solutions can be obtained in the following way. From (58) it is clear that we must not have ϕ_u be a function of $A(U + V)$. Again using (17) this would imply staticity. Therefore A is constrained by the following differential equation,

$$k^2 = 2 \frac{A' A''' + (2 + s) A'' A'^2 + (s + 1) A'^4 - A''^2}{A'' + A'^2} \quad (59)$$

where k is an arbitrary constant. Obviously for such A' we get implicitly a 2-dimensional solution with a linearly dependent dilaton field on $U - V$. Let us analyze here for simplicity the example $A = U + V$ [18]. With the coordinate transformation (17) the solution simplifies to,

$$ds^2 = \frac{1}{2\alpha} e^{(\gamma^2)z} e^{\gamma\eta\sqrt{\gamma^2 - \frac{2}{D-2}}t} (-dt^2 + dz^2) + e^{\frac{2z}{D-2}}$$

$$\phi = \phi_0 - \eta\sqrt{\gamma^2 - \frac{2}{D-2}}t - \gamma z$$

The first thing to note is that this solution is valid as long as $\gamma^2 > \frac{2}{D-2}$. Hence in $D = 10$ the critical point is $\gamma = 1/2$ where the solution is again static and belongs to Class I. For our cases of interest the solution is then well defined. Furthermore and most importantly this solution is everywhere regular. Just like the case of thick domain walls [26] or global vortices [27] we see that the solution is regular once we relax the staticity requirement. Indeed the above solution has the same general form as a thick planar domain wall solution [26] by taking the coordinate transformation,

$$U = \frac{1}{2(\gamma - \eta\sqrt{2(s+1)})}(z - t), \quad V = \frac{1}{2(\gamma + \eta\sqrt{2(s+1)})}(z + t)$$

where we have kept the labels for our coordinates as z and t . The solution reads,

$$ds^2 = e^{\gamma z} (-dt^2 + dz^2) + e^{\frac{2\eta t\sqrt{\gamma^2 - \frac{2}{D-2}}}{D-2}} e^{\gamma z} dx^8 \quad (60)$$

$$\phi = \phi_0 - z$$

We see that the scalar field is now static but spacetime is not. Note however the absence of the event horizon which characterises the fact that the scalar field [26] is a topological soliton.

There is the case of $\alpha = 0$ we can study completely following the work of Tabensky and Taub [43].

⁹The breakdown of Birkhoff's theorem in this context was mentioned in [17] where the authors refer to a forthcoming publication (see references within) discussing this.

A. The SUSY limit $\alpha = 0$

The case of $\alpha = 0$ amounts to switching off the Liouville potential in our action (2). This case was studied in 4 dimensions by Tabensky and Taub [43] for a stiff perfect fluid source (where pressure is equal to energy density). We sketch the extension to D dimensions here. Explicit examples can be found in the original paper as well as more recently in [44] for a T-dual version of the Sugimoto model [22].

The B equation (53a) with $\alpha = 0$ is just the two-dimensional wave equation which has general solution,

$$B = F(u) + G(v)$$

with F and G arbitrary functions. Substituting the solution of B into (53) yields,

$$(2\chi)_{uv} = -\frac{1}{2}\tilde{\phi}_u\tilde{\phi}_v \quad (61a)$$

$$\tilde{\phi}_{uv} = -\frac{1}{2(F+G)}(\tilde{\phi}_u G' + \tilde{\phi}_v F') \quad (61b)$$

$$(2\chi)_u - \frac{F''}{F'} = \frac{F+G}{2F'}\tilde{\phi}_u^2 \quad (61c)$$

$$(2\chi)_v - \frac{G''}{G'} = \frac{F+G}{2G'}\tilde{\phi}_v^2 \quad (61d)$$

Upon making the coordinate transformation,

$$(u, v) \rightarrow (F(u), G(v))$$

the field equations reduce to,

$$(2\chi)_{FG} = -\frac{1}{2}\tilde{\phi}_F\tilde{\phi}_G \quad (62a)$$

$$\tilde{\phi}_{FG} = -\frac{1}{2(F+G)}(\tilde{\phi}_F + \tilde{\phi}_G) \quad (62b)$$

$$(2\chi)_u - [ln(F')] = \frac{[(F+G)^2]'}{4}\tilde{\phi}_F^2 \quad (62c)$$

$$(2\chi)_v - [ln(G')] = \frac{[(F+G)^2]'}{4}\tilde{\phi}_G^2 \quad (62d)$$

Now (62b) is recognised as an Euler-Poisson-Darboux type equation (see for instance Ames 1965)

$$u_{xy} + \frac{N}{x+y}(u_x + u_y) = 0 \quad (63)$$

which has general solution,

$$u(x, y) = \frac{\partial^{N-1}}{\partial(x+y)^{N-1}} \left(\frac{\Phi(x) + \Psi(y)}{(x+y)^N} \right)$$

when N is a positive integer. In our case of interest, where $N = 1/2$, the equation can be interpreted as a 2 dimensional wave equation in cylindrical coordinates. This can be achieved by restoring time and space-like coordinates¹⁰,

$$F = r - t, \quad G = t + r$$

to get,

$$\tilde{\phi}_{rr} + \frac{1}{r}\tilde{\phi}_r - \tilde{\phi}_t = 0 \quad (64)$$

The general solution is given by means of a variety of integral representations (see Copson Partial differential equations for a beautiful derivation using complex analysis) here we choose Poisson's original formula,

$$\tilde{\phi}(t, r) = \int_0^\pi \Phi(t + r\cos\psi)d\psi + \int_0^\pi \Psi(t + r\cos\psi)\log(r\sin^2\psi)d\psi$$

with Φ and Ψ arbitrary C^2 functions.

Hence given Φ and Ψ we can find $\tilde{\phi}$ and then integrate once (62c) and (62d) in order to find the χ field.

Let us now consider the question of unicity of the field equations (9). The following argument based on the above analysis leads us to postulate that the one dimensional solutions (section III and IV) along with the 2 dimensional solutions (59) constitute the general solution to the field equations (9). From the general solution for the case of $\alpha = 0$ above we saw that component B verifies a two dimensional wave equation hence $B = U + V$. Note then that the Taub planar solutions in the vacuum [47] i.e. in the absence of a scalar field, also admit as general solution $B = U + V$ *in the same coordinate system*. On the other hand in the case of a bulk of constant curvature, a cosmological constant [34], the B component verifies a non-homogeneous wave equation and it turns out that $B = B(U + V)$ with B an arbitrary real function of $U + V$. We remind the reader that under (17) this simply means that B is a static field. In the case under consideration we showed that the field equations (9) could be transformed into (53), the case of a cosmological constant with a scalar field. Therefore by symmetry one is tempted to postulate that the general solution for B is likewise $B = B(U + V)$ just like in the absence of a cosmological constant. Under this assumption and noting coordinate transformations (17) we saw that either we would get the one dimensional solutions or the 2-dimensional solutions implicitly given by (59). A counterargument to the above would be that the symmetries appearing in the solutions with or without cosmological constant and with or without scalar field are related to the particular conformal system of coordinates and are not true symmetries of the ensemble of solutions in a coordinate independent way.

VII. CONCLUSIONS

Starting from a D dimensional spacetime admitting $D - 2$ planar symmetry we derived in section IV, V the general static or time-dependant solutions. Furthermore in section VI

¹⁰In the original analysis [43], the authors used the Riemann-Volterra method involving hypergeometric functions

we analysed the field equations and found implicitly a class of two dimensional solutions given by (59). An example of these has recently been given in [18]. Having generalised to D dimensions the general solution for $\alpha = 0$ (no Liouville potential) [43] we conjectured in the end of section VI that the solutions we found constitute the unique solutions to (9).

For the 1-dimensional case i.e. where the fields are locally static or time dependant we found three classes of solutions classified by the type of roots of a second degree polynomial ($s \neq 0$). Hence Class I solutions involved two distinct roots for $f(p)$, Class II one double root etc. The locations of the roots of $f(p)$ stood for candidate singularities of spacetime. We applied these solutions as gravitating backgrounds to $D = 10$ non-supersymmetric string theories, in particular the open Type I theory with gauge group $USp(32)$ and the heterotic theory with cosmological constant and gauge group $SO(16) \times SO(16)$. On passing we obtained the general $SO(9)$ maximal symmetry solutions recently discussed in [15]. We saw how the type I theory backgrounds, $s = 0$, involved a critical value for the gravitational field which restricted considerably the possible solutions since then $f(p)$ was linear. For example whereas the heterotic string admits 3 static and 1 time dependent solutions of maximal $SO(9)$ symmetry the open string admits solely 1 static and 1 time dependent solution.

All one dimensional solutions depended on two integration constants $c = 0, -1, 1$ and d . The former originated from the scalar field and was of a topological nature whereas the latter was related to Weyl curvature and $d = 0$ simply meant that we had a conformally flat spacetime. In the table above we have gathered our results for the static heterotic string solutions.

Static backgrounds for the non-susy heterotic string, $\gamma = 5/2$, $D = 10$						
	d	0 Class I	$d \leq -9/2$ Class I, II	$-9/2 < d < -1/2$	$-1/2 \leq d < 0$ Class I, II	$d > 0$ Class I
c		SO(9)	SO(8)			
0	<i>Sect. III</i>	Asymptotic Solution	Naked singularity $\infty \leftarrow R \rightarrow 0$			Black Hole $\infty \leftarrow R \rightarrow 0$
1	<i>Sect. IV</i>	infinite proper dist. $\infty \leftarrow R \rightarrow 0$	compact proper dist. $\infty \leftarrow R \rightarrow \infty$	compact proper dist. Class III $\infty \leftarrow R \rightarrow \infty$	infinite proper distance $\infty \leftarrow R \rightarrow 0$	
-1	<i>Sect. IV</i>	compact proper dist. $\infty \leftarrow R \rightarrow \infty$	infinite proper dist. $\infty \leftarrow R \rightarrow 0$		Compact proper dist. $\infty \leftarrow R \rightarrow \infty$	

The general characteristics of the 1-dimensional solutions are as follows: Firstly they are all singular. When $c = 0$ the singularity could sometimes be censored by a horizon yielding in the static case black hole solutions first studied by [40]. For $c = \pm 1$ in certain cases we found that proper distance in the independent spacelike variable was actually finite in agreement with the $SO(9)$ solutions of [15]. Then the topology of spacetime is an interval times a nine

dimensional manifold. Furthermore we found that compactness was equivalently related to singular behavior of the curvature tensor. Indeed spacetime is always singular at the endpoints of the compact proper interval. On the contrary when proper distance is infinite then we had a naked singularity at the origin with dilaton matter smoothly going to zero at large distances or late times. Hence when proper distance is infinite the dilaton field will always roll down the potential well towards minus infinity where the Liouville potential acquires its global minimum. We can actually assert that if the dilaton does not roll to its vacuum then proper distance has to be compact. Interestingly the compact $SO(9)$ solutions discussed in detail in [15] have effectively not only a 9-dimensional behavior but also the dilaton rolls to minus infinity at one of the endpoints of the interval. Hence one anticipates that this solution should be classically stable against one of the remaining $SO(9)$ solutions. Another argument in favor of its stability is that the same type of solution appears for the Type I string and there it is the unique static $SO(9)$ solution [15].

We should also stress the fact that none of the solutions we found are asymptotically flat or of constant curvature. This seems surprising. Indeed as we noted above the scalar field typically will roll to the global minimum of the potential. However following the works of Poletti and Wiltshire for dilatonic black holes [29] the non-asymptotic flatness is related to the fact that the potential under question does not acquire its minimum at a finite value for ϕ . Hence even if we allow the dilaton to reach its vacuum value, Minkowsky or constant curvature spacetime is not a gravitational background for non-supersymmetric string theories at least at the classical level we are considering.

Let us now turn to the singular nature of the 1-dimensional solutions. It is known from studies of gravitating topological defects [24], [25] that the singular nature of a thick domain wall [26] or of a global vortex [27] is due to the fact that we impose a static spacetime. Indeed on allowing spacetime to be space and time dependent global vortex and thick domain wall spacetimes are everywhere regular. Exactly the same thing occurs here. We saw that a two dimensional solution [18] is everywhere regular and can be transformed in such a way to acquire the form of a thick domain wall solution (60). By this we mean that the scalar field is static with the spacetime metric having an exponential time dependence and a conformal space dependent factor. However here we have no horizon since the potential is not of a domain wall type, acquiring a non degenerate discrete set of minima.

The solutions we have obtained can be used as background solutions incorporating the motion of brane Universe type wall much in a generalized context of [7] (for a recent discussion with a two dimensional background see [18]). Indeed these solutions are relevant to cosmological perturbations, brane cosmology in a non-constant curvature background and also to the radion related issues [48]. This setup is a generalization of a constant curvature spacetime since a Liouville potential closely resembles a cosmological constant. In this case we can indeed picture the bulk as a fluid of dark matter flowing through the brane Universe. Thus comes about the question of Birkhoff's theorem and its relevance to brane cosmology. It was proven recently that Birkhoff's theorem applies in the case of a spacetime of constant curvature admitting a $D - 2$ spherical planar or hyperboloidal symmetry [34]. The unicity then implies Kottler's solution (topological black hole) [36] as the unique brane cosmology background. To understand the essence of this let us step back to usual 4-dimensional cosmology. On solving the FLRW equations one finds one physical degree of freedom, the expansion rate of the Universe which is related to energy density and pressure. When one

considers brane cosmology in 5 dimensions our Universe is a timelike hypersurface evolving in a 5-dimensional spacetime. With the addition of one dimension one would naively expect that we would obtain another physical degree of freedom from the field equations. However Birkhoff's theorem does not allow this. Indeed two dimensional conformal symmetries ensure that we have an extra Killing vector which then implies only one physical degree of freedom, the wall trajectory, alias the expansion rate as witnessed by a four dimensional observer. However as we saw scalar field matter breaks this unicity theorem and two dimensional solutions exist. This fact is independent of the presence of the potential indeed a potential rather restricts the possible solutions rather than enhances them. Could there exist potentials so that an extension to Birkhoff's theorem would hold? This seems unlikely since a Liouville type potential is a natural generalization of a cosmological constant.

Let us end on higher order corrections from the coupling constant expansion. For instance in the case of the Sugimoto model [22] and also for the Sagnotti [21] non supersymmetric string model the potential to consider at one loop and in the Einstein frame is $V(\phi) = \alpha_1 e^{3\phi/2} + \alpha_2 e^{5\phi/2}$ with α_1 and α_2 positive constants. Note now the interplay between the critical value $\gamma = 3/2$ and $\gamma = 5/2$. Firstly it is simple to show that for α_1 and α_2 positive no maximally symmetric solution exists. Secondly on a brief analysis of the field equations it is not evident that even $SO(9)$ solutions exist with such a potential. We hope to be reporting progress on this and further issues in the near future.

VIII. ACKNOWLEDGMENTS

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